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Principles of Distributed Computing
Exercise 7: Sample Solution

1 Pancake Networks

Generally, observe that $N = |V(P_n)| = n! \in O(n^n) \Rightarrow n \in O(\frac{\log N}{\log \log N})$.

a) See Figure 1. For drawing P_n, first draw n copies of P_{n-1}, each of which will have some $j \in [n]$ fixed as the last vertex. Then there are $(n-2)!$ nodes of such a P_{n-1} connected to the same $(n-1)$-dimensional pancake. To see this, fix v_1 and v_n, the remaining node combinations in the middle will be the link between pancake $P_{n-1}|v_n$ and $P_{n-1}|v_1$. There are $n-1$ such sets in $P_{n-1}|v_n$, each connecting with another $(n-1)$-dimensional pancake.

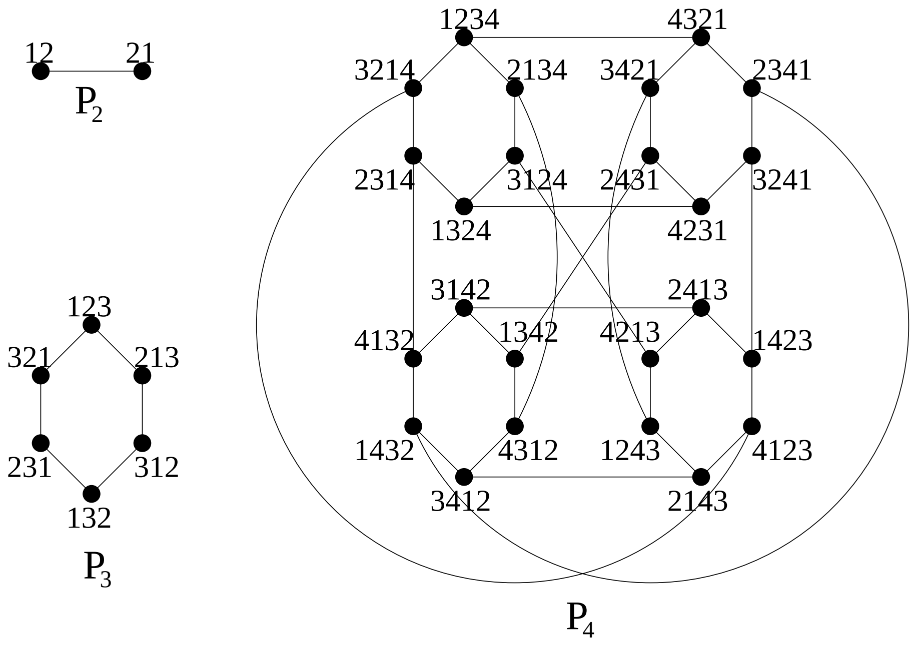

Figure 1: Pancake graphs for $n = 2, 3, 4$.

b) Let us look at the second, more intuitive definition (Eq. (3)). Basically, it states that for every node, there exists exactly one edge for every distinct prefix reversal. So the node degree of P_n can be stated as follows: how many non-trivial prefix reversals are there for a sequence of n nodes? Answer: $n-1$ with edges $e_2, \ldots, e_n$. Succinctly,

$$deg(v) = n - 1 \qquad \forall v \in V(P_n).$$

Thus the degree of an N-node pancake graph is in $O(\log N / \log \log N)$.

c) To give an upper bound on the diameter, we need to determine in how many steps, at most, we can go from one node to any other node. Say we want to get from node $v = v_1v_2\dots v_n$ to node $w = w_1w_2\dots w_n$. As with all hypercube-like graphs, we will proceed by correcting one "coordinate" at a time. In this case, we start at the back. Since the nodes are all permutations, there will exist a v_j such that $v_j = w_n$. Now take the edges $v \to e_j \to e_n$ to get to node $v^{(1)} = v_N \dots v_{j+1}v_1v_2\dots v_{j-1}w_n$. We can relable the indices of $v^{(1)}$ to go again from 1 to $n-1$, leave w_n fixed, find the index j with $v_j = w_{n-1}$, and take the edges $v^{(1)} \to e_j \to e_{n-1}$. Thus, by induction, we need at most 2 edges per correct target index, and we are done after $n-1$ steps. Therefore,

$$D(P_n) \leq 2(n-1)$$

that is, the diameter of P_n is in $O(\log N/\log\log N)$.

Gates and Papadimitriou [1] have also shown that this is asymptotically optimal, that is,

$$D(P_n) \geq n.$$

d) To show that P_n is Hamiltonian, we proceed by induction on n. We will actually show the following stronger claim: In P_n, there exists a Hamiltonian path from $12\dots(n-1)n$ to $n(n-1)\dots21$ and the cycle is completed by using edge e_n. Observe that since in P_n the graph looks the same from every vertex, this also holds for any given vertex $v_1v_2\dots v_n$.

For $n=3$: by direct observation, we have the path $123 \to 213 \to 312 \to 132 \to 231 \to 321$ and the final edge $321 \to 123$.

Assume that P_{n-1} has such a Hamiltonian path H_{n-1} from $v_1v_2\dots v_{n-1}$ to $v_{n-1}\dots v_2v_1$. Then we can construct a Hamiltonian path in P_n by concatenating the Hamiltonian paths of the n P_{n-1} subgraphs as follows:

$$\begin{gathered}
a_n = 12\dots(n-1)n \to H_{n-1} \to (n-1)\dots21n = b_n \\
b_n \to e_n \to a_{n-1} \\
a_{n-1} = n12\dots(n-2)(n-1) \to H_{n-1} \to n-2\dots1n(n-1) = b_{n-1} \\
b_{n-1} \to e_n \to a_{n-2} \\
\vdots \\
a_2 = 3\dots(n-1)n12 \to H_{n-1} \to 1n(n-1)\dots32 = b_2 \\
b_2 \to e_n \to a_1 \\
a_1 = 2\dots(n-1)n1 \to H_{n-1} \to n(n-1)\dots21 = b_1
\end{gathered}$$

and we complete the cycle with the final $b_1 \to a_n$ edge. Or, more formally, set

$$\begin{aligned}
a_i &= (i+1)(i+2)\dots n1\dots(i-1)i \\
b_i &= (i-1)\dots1n\dots(i+2)(i+1)i
\end{aligned}$$

using $n+1=1$ and $1-1=n$. Then, since the nth coordinate is fixed, the Hamiltonian path H_{n-1} from a_i to b_i is completely contained in $n-1$ dimensions. Its existence is guaranteed by the induction hypothesis. Thus, the Hamiltonian path in n dimensions is given by

$$a_n \overset{H_{n-1}}{\cdots} b_n \to a_{n-1} \overset{H_{n-1}}{\cdots} b_{n-1} \to a_{n-2}\dots a_2 \overset{H_{n-1}}{\cdots} b_2 \to a_1 \overset{H_{n-1}}{\cdots} b_1 \to a_n$$

where $a_n = 12\dots(n-1)n$ and $b_1 = n(n-1)\dots21$ as required in the claim.

References

[1] W. H. Gates, C. H. Papadimitriou, Bounds for sorting by prefix reversal, Discrete Math. 27, (1979), 47–57.